On two claims about the closedness of ideals in [Wei10]

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We discuss two claims made in Weinstein's paper [Wei10] about the closedness of principal ideals that are used in his proof of Theorem 2.7.3. We use the exact same notation as in this paper, which we therefore do not explain here.

Closedness of ηA, where $\eta \in A_r$ for some r

The fact that $\eta_r A$ is closed, is used in the proof of [Wei10][Lemma 2.8.13]. We prove more generally that ηA is closed whenever $\eta \in A_r$ for some finite $r \geq 0$. We recall the following lemma comparing the maximal ideals of A_1 and A_m:

Lemma 1 ([Wei10][Lemma 2.8.11])**.** *Let $n \geq 1$. As ideals in A_m we have $IA_m \subset I_m^{[q^{n(m-1)}]} \subset I_m^{q^{n(m-1)}}$ and $I_m^{nq^{n(m-1)}} \subset IA_m$.*

The proof of closedness of ηA relies on the above lemma and the fact that the A_m are regular local rings.

Lemma 2. *Let $\eta \in A_r$. Then ηA is closed.*

Proof. If η is 0 in A, this is trivial. So suppose η is non-zero. Let $\ell \gg 0$ such that $\eta \notin I^\ell A$.
Let $N \geq 1$ and suppose $\eta x \in I^N A$. Write $\widetilde{A} \coloneqq \varinjlim_m A_m$, which becomes A when I-adically completed. By [Sta22, Tag 05GG] the canonical map

$$\widetilde{A}/I^N\widetilde{A} \to A/I^N A$$

is an isomorphism. Therefore, there exists an $m \geq r$ such that we can choose an $\tilde{x} \in A_m$ with $x - \tilde{x} \in I^N A_m$ and $\eta\tilde{x} \in I^N A_m$. By Lemma 1 this means

$$\eta\tilde{x} \in I_m^{Nq^{n(m-1)}}.$$

Since A_m is a regular local ring, its associated graded ring $\oplus_{i\geq 0} I_m^i/I_m^{i+1}$ is isomorphic as a graded k-algebra to a polynomial ring over k (in $\dim A_m = h$ variables). By Lemma 1, $\eta \notin I_m^{\ell n q^{n(m-1)}}$, we see, using the graded isomorphism of the associated graded ring with $k[T_1, \ldots, T_n]$, that

$$\tilde{x} \in I_m^{(N-\ell n)q^{n(m-1)}} \subset I^{\lfloor \frac{N}{n} - \ell \rfloor} A_m,$$

where we have used Lemma 1 for the inclusion again and where we write $\lfloor\cdot\rfloor$ for the floor function. We conclude that

$$x \in I^{\lfloor \frac{N}{n} - \ell \rfloor} A. \tag{1}$$

Therefore, if $\eta x_1, \eta x_2, \dots$ is a sequence in ηA that has a limit in A, then this limit is in ηA: indeed, let $N \gg 0$, then there is an M such that for all $i, j > M$ we have $\eta(x_i - x_j) \in I^N A$. Then by 1 we have $x_i - x_j \in I^{\lfloor \frac{N}{n} - \ell \rfloor} A$, which proves the sequence $x_1, x_2, \dots$ also converges, to x, say. Then necessarily $\lim_{i\to\infty} \eta x_i = \eta x \in \eta A$. This proves closedness of ηA. ■

Closedness of πB

Recall that

$$B = \mathcal{O}_{K_0}[\![X_1^{1/q^\infty}, \dots, X_n^{q^\infty}]\!] \widehat{\otimes}_{\mathcal{O}_{K_0}[\![X^{1/q^\infty}]\!]} \mathcal{O}_{\hat{K}_\infty},$$

that is, B is the $(\pi, X_1 \otimes 1, \dots, X_n \otimes 1)$-adic completion of

$$\widetilde{B} \coloneqq \mathcal{O}_{K_0}[\![X_1^{1/q^\infty}, \dots, X_n^{q^\infty}]\!] \otimes_{\mathcal{O}_{K_0}[\![X^{1/q^\infty}]\!]} \mathcal{O}_{\hat{K}_\infty}.$$

Recall that the map $\mathcal{O}_{K_0}[\![X^{1/q^\infty}]\!] \to \mathcal{O}_{K_0}[\![X_1^{1/q^\infty}, \dots, X_n^{1/q^\infty}]\!]$ sends X^{1/q^m} to $\delta(X_1, \dots, X_n)^{1/q^m}$ and that $\mathcal{O}_{K_0}[\![X^{1/q^\infty}]\!] \to \mathcal{O}_{\hat{K}_\infty}$ sends X^{1/q^m} to

$$\varpi^{1/q^m} = \lim_{i\to\infty} \left(X^{(m+i)}\right)^{q^i},$$

where $\left(X^{(1)}, X^{(2)}, \dots\right) \in \widetilde{\Lambda^n G}(\mathcal{O}_{\hat{K}_\infty})$ with each $X^{(i)}$ a (universal) Drinfeld basis for $\Lambda^n G[\pi^i](\mathcal{O}_{\hat{K}_\infty})$. The $X^{(i)}$ topologically generate $\mathcal{O}_{\hat{K}_\infty}$ over $\mathcal{O}_{K_0}$ and can be expressed as

$$X^{(i)} = \lim_{m\to\infty} [\pi^m]_{\Lambda^n G}\left(\varpi^{1/q^{m+i}}\right).$$

It follows that the map $\mathcal{O}_{K_0}[\![X^{1/q^\infty}]\!] \to \mathcal{O}_{\hat{K}_\infty}$ is surjective, because the ϖ^{1/q^m} topologically generate $\mathcal{O}_{\hat{K}_\infty}$ over $\mathcal{O}_{K_0}$.
The morphism

$$\mathcal{O}_{K_0}[\![X_1^{1/q^\infty}, \dots, X_n^{1/q^\infty}]\!] \to \widetilde{B}$$

is therefore also surjective and its kernel $\tilde{\mathfrak{b}}$ is generated by the relations

$$\text{“}\delta(X_1, \dots, X_n)^{1/q^m} = \varpi^{1/q^m}\text{”}.$$

Since $\mathcal{O}_{K_0}[\![X_1^{1/q^\infty}, \dots, X_n^{1/q^\infty}]\!]$ is $(\pi, X_1, \dots, X_n)$-adically complete, it follows that the morphism $\mathcal{O}_{K_0}[\![X_1^{1/q^\infty}, \dots X_n^{1/q^\infty}]\!] \to B$ is surjective with the closure $\mathfrak{b}$ of $\tilde{\mathfrak{b}}$ as kernel.
The ideal $\pi\mathcal{O}_{K_0}[\![X_1^{1/q^\infty}, \dots, X_n^{1/q^\infty}]\!]$ is clearly closed and one might hope to reduce to this, but unfortunately this is not as easily done as (perhaps) thought. Indeed, closedness of πB is equivalent to closedness of the ideal

$$\mathfrak{b} + \pi\mathcal{O}_{K_0}[\![X_1^{1/q^\infty}, \dots, X_n^{1/q^\infty}]\!],$$

because closedness of πB is equivalent to separatedness of $B/\pi B$.
From this, closedness of πB can be seen equivalent to closedness of the image of $\mathfrak{b}$ in

$$\mathcal{O}_{K_0}[\![X_1^{1/q^\infty}, \dots, X_n^{1/q^\infty}]\!] / \pi\mathcal{O}_{K_0}[\![X_1^{1/q^\infty}, \dots, X_n^{1/q^\infty}]\!] \cong \overline{\mathbb{F}_q}[\![X_1^{1/q^\infty}, \dots, X_n^{1/q^\infty}]\!].$$

We claim that actually the image of $\tilde{\mathfrak{b}}$ is closed in $\overline{\mathbb{F}_q}[\![X_1^{1/q^\infty},\dots,X_n^{1/q^\infty}]\!]$ already and thus this image must be the same as $\mathfrak{b}$, which is therefore closed, proving closedness of πB.
Indeed, the quotient of $\overline{\mathbb{F}_q}[\![X_1^{1/q^\infty},\dots,X_n^{1/q^\infty}]\!]$ by the image of $\tilde{\mathfrak{b}}$ is

$$\begin{aligned}\overline{\mathbb{F}_q}[\![X_1^{1/q^\infty},\dots,X_n^{1/q^\infty}]\!]\otimes_{\overline{\mathbb{F}_q}[\![X^{1/q^\infty}]\!]}\mathcal{O}_{\hat{K}_\infty}/\pi &\cong \overline{\mathbb{F}_q}[\![X_1^{1/q^\infty},\dots,X_n^{1/q^\infty}]\!]\otimes_{\overline{\mathbb{F}_q}[\![X^{1/q^\infty}]\!]}\mathcal{O}_{K_\infty}/\pi\\ &\cong \overline{\mathbb{F}_q}[\![X_1^{1/q^\infty},\dots,X_n^{1/q^\infty}]\!]\otimes_{\overline{\mathbb{F}_q}[\![X^{1/q^\infty}]\!]}\left(\varinjlim_m \mathcal{O}_{K_m}/\pi\right).\end{aligned}$$

Now notice that $\varpi^{1/q^m}\equiv X^{(m)} \mod \pi$, which follows from $[\varpi]_{\Lambda^n G}(X)\equiv X^q \mod \pi$. Therefore, we see that this quotient is isomorphic to

$$\begin{aligned}&\varinjlim_m\left(\overline{\mathbb{F}_q}[\![X_1^{1/q^\infty},\dots,X_n^{1/q^\infty}]\!]\otimes_{\overline{\mathbb{F}_q}[\![X^{1/q^m}]\!]}\mathcal{O}_{K_m}/\pi\right)\cong\\ &\varinjlim_m \overline{\mathbb{F}_q}[\![X_1^{1/q^\infty},\dots,X_n^{1/q^\infty}]\!]/\left(\Phi_m\left(\delta(X_1,\dots,X_n)^{1/q^m}\right)\right)=\\ &\varinjlim_m \overline{\mathbb{F}_q}[\![X_1^{1/q^\infty},\dots,X_n^{1/q^\infty}]\!]/\left(\left(\delta(X_1,\dots,X_n)^{1/q^m}\right)^{(q-1)q^{m-1}}\right)=\\ &\varinjlim_m \overline{\mathbb{F}_q}[\![X_1^{1/q^\infty},\dots,X_n^{1/q^\infty}]\!]/\left(\left(\delta(X_1,\dots,X_n)^{1/q}\right)^{q-1}\right)\cong\\ &\overline{\mathbb{F}_q}[\![X_1^{1/q^\infty},\dots,X_n^{1/q^\infty}]\!]/\left(\left(\delta(X_1,\dots,X_n)^{1/q}\right)^{q-1}\right).\end{aligned}$$

Closedness of the image of $\tilde{\mathfrak{b}}$ is equivalent to separatedness of the quotient, which we see is equivalent to closedness of the principal ideal of $\overline{\mathbb{F}_q}[\![X_1^{1/q^\infty},\dots,X_n^{1/q^\infty}]\!]$ generated by $\left(\delta(X_1,\dots,X_n)^{1/q}\right)^{q-1}$. A moment's thought shows that the latter ideal is indeed closed, whence we conclude. ■

References

[Wei10] Jared Weinstein. *Semistable models for modular curves of arbitrary level*. 2010. DOI: 10.48550/ARXIV.1010.4241. URL: https://arxiv.org/abs/1010.4241.

[Sta22] The Stacks project authors. *The Stacks project*. https://stacks.math.columbia.edu. 2022.