Gelfand-Kirillov bound for GL_2

Reinier Sorgdrager

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Main theorem 1

Let p be a prime number, and L and K be two p-adic fields.

Theorem (S.)

Let Π be an admissible L-Banach representation of $\mathrm{GL}_2 K$ (or $D^\times$) such that Π^{la} has an infinitesimal character. Then Π has Gelfand-Kirillov dimension $\leq [K : \mathbf{Q}_p]$.

Remark

- *One can replace $\mathrm{GL}_2 K$ by $D^\times$, where D is the quaternion algebra over K.*
- *$< 2[K : \mathbf{Q}_p]$ was obtained by Dospinescu–Paškūnas–Schraen.*
- *Our bound is optimal (equality for principal series).*

Plan of talk

1. Main theorem 1 and explanation
2. p-adic Langlands
3. Main theorem in families and applications

p-adic representations

Let ϖ_L be a uniformizer of L. Write $k_L := \mathcal{O}_L/\varpi_L$.
Let G be a p-adic Lie group. We study

- *smooth mod p representations of G*: rep's of G on discrete k_L-vector spaces V with $G \times V \to V$ continuous,
- *L-Banach representations of G*: rep's of G on L-Banach spaces Π with $G \times \Pi \to \Pi$ continuous.

Example

The space of continuous functions $C(G, k_L)$ is a smooth k_L-rep.
If G is compact, $C(G, L)$ is an L-Banach representation (with the sup-norm).

Admissibility

Definition

A Banach or mod p rep. V is called *admissible* if

$$V|_H \hookrightarrow C(H, \text{coeff's})^{\oplus d_H}$$

for some $d_H \in \mathbf{Z}_{\geq 0}$ and some compact open subgroup H of G (equivalently any such H).

Duality and Iwasawa algebras

Mod p dual: $(-)^\vee := \mathrm{Hom}_{k_L}^{\mathrm{cont}}(-, k_L)$.

Schneider-Teitelbaum's dual: $(-)^* := \mathrm{Hom}_{L}^{\mathrm{cont}}(-, L)$.

$$C(H, k_L)^\vee = \left(\varinjlim_{\substack{N \lhd H \\ \text{cpt open}}} C(H/N, k_L) \right)^\vee \cong \varprojlim_{\substack{N \lhd H \\ \text{cpt open}}} C(H/N, k_L)^\vee$$

$$\cong \varprojlim_{\substack{N \lhd H \\ \text{cpt open}}} k_L[H/N]$$

$$=: k_L[\![H]\!]$$

"mod p Iwasawa algebra" of H

Duality and Iwasawa algebras

$$\mathcal{O}_L[\![H]\!] := \varprojlim_{\substack{N \lhd H \\ \text{cpt open}}} \mathcal{O}_L[H/N]$$

$$L[\![H]\!] := \mathcal{O}_L[\![H]\!][1/\varpi_L], k_L[\![H]\!] = \mathcal{O}_L[\![H]\!]/\varpi_L$$

Example

$H = \mathbf{Z}_p^s$, $s \in \mathbf{Z}_{\geq 0}$:

$$\mathcal{O}_L[\![\mathbf{Z}_p^s]\!] \cong \mathcal{O}_L[\![X_1, \ldots, X_s]\!], \text{ but } L[\![\mathbf{Z}_p^s]\!] \ncong \mathcal{O}_L[\![X_1, \ldots, X_s]\!].$$

Duality and Iwasawa algebras

V smooth admissible k_L-rep. of H:

$$V \hookrightarrow C(H, k_L)^{\oplus d}$$

dualizes to

$$k_L[\![H]\!]^{\oplus d} \twoheadrightarrow V^\vee.$$

$(-)^\vee$ induces an anti-equivalence:

$$\left\{\begin{array}{c}\text{admissible smooth}\\ k_L\text{-representations of } H\end{array}\right\} \xrightarrow{\sim} \left\{\begin{array}{c}\text{finitely generated}\\ k_L[\![H]\!]\text{-modules}\end{array}\right\}^{\mathrm{op}}.$$

And $(-)^*$ in the Banach case:

$$\left\{\begin{array}{c}\text{admissible } L\text{-Banach}\\ \text{representations of } H\end{array}\right\} \xrightarrow{\sim} \left\{\begin{array}{c}\text{finitely generated}\\ L[\![H]\!]\text{-modules}\end{array}\right\}^{\mathrm{op}}.$$

Locally analytic vectors

$$C(H,L) \supset C^{\mathrm{la}}(H,L) := \{f : H \to L : \text{ locally a convergent power series}\}.$$

If

$$\Pi \subset C(H,L)^{\oplus d} \tag{1}$$

is admissible, then

$$\Pi^{\mathrm{la}} := \Pi \cap C^{\mathrm{la}}(H,L)^{\oplus d}$$

are the *locally analytic vectors* of Π. It is a representation of H independent of choice of (1).

Locally analytic vectors

$$D(H, L) := C^{\mathrm{la}}(H, L)^*$$

is the *locally analytic distribution algebra* of H.

$$C^{\mathrm{la}}(H, L) \subset C(H, L)$$

dualizes to L-alg. morphism

$$L[\![H]\!] \hookrightarrow D(H, L)$$

and we recover

$$\left(\Pi^{\mathrm{la}}\right)^* = D(H, L) \otimes_{L[\![H]\!]} \Pi^*.$$

Infinitesimal action

$$\mathrm{Lie}(G)\,\mathrm{Lie}(G) \otimes_{\mathbf{Q}_p} LU_L\left(\mathrm{Lie}(G) \otimes_{\mathbf{Q}_p} L\right) \curvearrowright \Pi^{\mathrm{la}}$$

via

$$\mathfrak{r}.v := \frac{d}{dt}\exp(t\mathfrak{r})v|_{t=0}$$

for $\mathfrak{r} \in \mathrm{Lie}(G) = T_1 G$, $v \in \Pi^{\mathrm{la}}$.
This action is given by an explicit map

$$U_L\left(\mathrm{Lie}(G) \otimes_{\mathbf{Q}_p} L\right) \hookrightarrow D(H, L).$$

Infinitesimal action

Definition

Let

$$\lambda \colon \mathcal{Z}\left(U_L\left(\mathrm{Lie}(G) \otimes_{\mathbf{Q}_p} L\right)\right) \to L$$

be a morphism of L-algebras.
We say Π has *infinitesimal character* λ if $\mathcal{Z}(\ldots)$ acts on

$$\Pi^{\mathrm{la}} = \left(D(H, L) \otimes_{L[\![H]\!]} \Pi^*\right)^*$$

via λ.

Main theorem 1

Recall:

Theorem (S.)

Let Π *be an admissible L-Banach representation of* $\mathrm{GL}_2 K$ *(or* $D^\times$*) such that* Π^{la} *has an infinitesimal character. Then* Π *has Gelfand-Kirillov dimension* $\leq [K : \mathbf{Q}_p]$.

Gelfand-Kirillov dimension

Morally: "**Krull dimension** of Π^* as a $L[\![H]\!]$-module".

Example

When H is commutative (say $H = \mathbf{Z}_p^s$), then

$$\mathrm{GKdim}(\Pi^*) = \mathrm{Krulldim}(\Pi^*) := \mathrm{Krulldim}\left(L[\![H]\!]/\operatorname{Ann}(\Pi^*)\right).$$

Gelfand-Kirillov dimension

Let $A \in \{k_L[\![H]\!], L[\![H]\!], D(H,L)\}$ and M and A-module.

$$\mathrm{GKdim} M := \dim H - \inf\left\{i\colon \mathrm{Ext}^i_A(M,A) \neq 0\right\}.$$

- $\dim H :=$ dim of H as $\mathbf{Q}_p$-manifold $=$ global dimension of A.
- $\inf\left\{i\colon \mathrm{Ext}^i_A(M,A) \neq 0\right\}$ is the *grade* of the A-module M.
- *Key*: let M be an f.g. p-torsion free $\mathcal{O}_L[\![H]\!]$-module, then

$$\begin{aligned}\mathrm{GKdim}(M/\varpi_L M) &= \mathrm{GKdim}(M[1/\varpi_L]) \\ &= \mathrm{GKdim}\left(D(H,L) \otimes_{\mathcal{O}_L[\![H]\!]} M\right).\end{aligned}$$

Heuristics for bounding the GK-dim

Simplest case: $\mathrm{GL}_2\,\mathbf{Q}_p$ and $\mathbf{Q}_p$-Banach Π with *trivial* infinitesimal character.

$$\mathfrak{gl}_2\mathbf{Q}_p = \langle e, f, h, z\rangle_{\mathbf{Q}_p},$$

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\mathcal{Z}\left(U(\mathfrak{gl}_2\,\mathbf{Q}_p)\right) = \mathbf{Q}_p[z, \Delta] \text{ polynomial alg.},$$

$$\Delta = \frac{1}{2}h^2 + ef + fe.$$

Trivial inf. char. means $z, \Delta \in D(\mathrm{GL}_2\,\mathbf{Q}_p, \mathbf{Q}_p)$ annihilate

$$(\Pi^{\mathrm{la}})^* = D(\mathrm{GL}_2\,\mathbf{Q}_p, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p[\![\mathrm{GL}_2\,\mathbf{Q}_p]\!]} \Pi^*.$$

Heuristics for bounding the GK-dim

Trivial inf. char. means $z, \Delta \in D(\mathrm{GL}_2 \mathbf{Q}_p, \mathbf{Q}_p)$ annihilate

$$(\Pi^{\mathrm{la}})^* = D(\mathrm{GL}_2 \mathbf{Q}_p, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p[\![\mathrm{GL}_2 \mathbf{Q}_p]\!]} \Pi^*.$$

This only gives **two** elements annihilating $(\Pi^{\mathrm{la}})^*$, so that

$$\begin{aligned}\mathrm{GKdim}(\Pi) &= \mathrm{GKdim}((\Pi^{\mathrm{la}})^*) \\ &= \text{“}\mathrm{Krulldim}\left(D(\mathrm{GL}_2 \mathbf{Q}_p, \mathbf{Q}_p)/\operatorname{Ann}(\Pi^{\mathrm{la}})^*\right)\text{”} \leq 4 - 2 = 2.\end{aligned}$$

- The bound of DPS (and ours) gives $\leq 2[\mathbf{Q}_p : \mathbf{Q}_p] - 1 = 1$, so we are missing an annihilating element.
- In general, $\leq 2[K : \mathbf{Q}_p]$ is the “easy” bound. DPS already showed something more is going on.

What actually (roughly) happens (for experts):

- we pick a suitable p-valued subgroup $H \subset \mathrm{GL}_2 K$;
- then $\mathrm{GKdim}(\Pi) = \mathrm{Krulldim}\,(\mathrm{gr}_r(L[\![H]\!])/\,\mathrm{Ann}\,(\mathrm{gr}_r \Pi^*))$ for the filtrations of Schneider-Teitelbaum and $r \in [1/p, 1)$;
- we exhibit $3[K : \mathbf{Q}_p]$ elements in $\mathrm{Ann}\,(\mathrm{gr}_r \Pi^*)$ for r close to 1, independent of Π, yielding

$$\mathrm{GKdim}(\Pi) \leq 4[K : \mathbf{Q}_p] - 3[K : \mathbf{Q}_p] = [K : \mathbf{Q}_p].$$

- For $K = \mathbf{Q}_p$, Δ gives us **two** annihilating elements:

$$\mathrm{gr}(p^2\Delta) = \frac{1}{2}\underbrace{\mathrm{gr}(ph)^2}_{\#1} + 2\varepsilon\underbrace{(\mathrm{gr}(e)\,\mathrm{gr}(pf))}_{\#2}, \text{ with } \varepsilon = \mathrm{gr}(p).$$

- If K is unramified one gets three such elements for each $K \hookrightarrow \overline{\mathbf{Q}}_p$, giving $3[K : \mathbf{Q}_p]$.

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mod p and p-adic Langlands

Aim: establish some kind of correspondence

$$\left\{\begin{array}{c} n\text{-dimensional continuous} \\ p\text{-adic rep.s of } \mathrm{Gal}(\overline{\mathbf{Q}}_p/K) \end{array}\right\} \overset{?}{\longleftrightarrow} \left\{\begin{array}{c} \text{unitary } p\text{-adic Banach} \\ \text{rep.s of } \mathrm{GL}_n K \end{array}\right\}.$$

- $n = 1$: class field theory
- $\mathrm{GL}_2 \mathbf{Q}_p$: Colmez, Breuil, Paškūnas, Dospinescu, ...
- Beyond: ???

Example

Mod p correspondence for $\mathrm{GL}_2 \mathbf{Q}_p$:

$$\left\{\begin{array}{c} \text{irreducible 2-dimensional} \\ \text{rep.s of } \mathrm{Gal}_{\mathbf{Q}_p} \text{ over } \overline{\mathbf{F}}_p \end{array}\right\}_{/\cong} \overset{1:1}{\longleftrightarrow} \left\{\begin{array}{c} \text{supersingular rep.s} \\ \text{of } \mathrm{GL}_2 \mathbf{Q}_p \text{ over } \overline{\mathbf{F}}_p \end{array}\right\}_{/\cong}.$$

Global strategy for $\mathrm{GL}_2 K$

Let $\overline{r}\colon \mathrm{Gal}_K \to \mathrm{GL}_2 k_L$ be a continuous Galois representation.
Want: a smooth k_L-rep. π of $\mathrm{GL}_2 K$.

- Choose a totally real number field F with $v|p$ and $F_v \cong K$.
- Choose a globalization $\overline{\rho}\colon \mathrm{Gal}_F \to \mathrm{GL}_2 k_L$, i.e. $\overline{\rho}|_{\mathrm{Gal}_K} \cong \overline{r}$.
- Choose a suitable Shimura curve $X_{U^v U_v}$ over F with fixed level U^v away from v, varying $U_v \subset \mathrm{GL}_2 K$ cpt open subgroups.
- Set $\pi := \varinjlim_{U_v} \mathrm{Hom}_{\mathrm{Gal}_F}\left(\overline{\rho}, H^1_{\text{ét}}(X_{U^v U_v}, k_L)\right)$, a smooth admissible k_L-rep. of $\mathrm{GL}_2 K$.
- Motivation: Emerton's local-global compatibility result for $\mathrm{GL}_2 \mathbf{Q}_p$.
- Problem: is π independent of the choices?

Patching (Caraiani–Emerton–Gee–Geraghty–Paškūnas–Shin)

- CEGGPS propose even more choices to get candidates for a p-adic correspondence: patching.
- $R^{\square}_{\overline{r}}$ framed Galois deformation ring of $\overline{r}$ (over $\mathcal{O}_L$).
- $(R_\infty, \mathfrak{m}_\infty)$ complete local Noetherian $\mathcal{O}_L$-algebra $\supset R^{\square}_{\overline{r}}$.
- $\mathbb{M}_\infty$ "patched module": a finitely generated p-torsion free $R_\infty[\![\mathrm{GL}_2 K]\!]$-module. $\mathbb{M}_\infty/\mathfrak{m}_\infty = \pi^\vee$.

Patching (CEGGPS)

- $\mathbb{M}_\infty$ "patched module": a finitely generated p-torsion free $R_\infty[\![\mathrm{GL}_2 K]\!]$-module. $\mathbb{M}_\infty/\mathfrak{m}_\infty = \pi^\vee$. $R^{\square}_{\overline{r}} \subset R_\infty$.

Take $y \in \mathrm{SpecMax}\, R_\infty[1/p]$. Residue field L_y is a p-adic field. This gives a p-adic lift

$$\rho_y \colon \mathrm{Gal}_K \to \mathrm{GL}_2(L_y)$$

of $\overline{r}$, via $R^{\square}_{\overline{r}} \to R_\infty[1/p] \twoheadrightarrow L_y$.

Conjecture (CEGGPS)

Take $y \in \mathrm{SpecMax}\, R_\infty[1/p]$. The unitary admissible Banach representation of $\mathrm{GL}_2 K$ given by

$$\Pi_y := \mathrm{Hom}^{\mathrm{cont}}_L(\mathbb{M}_\infty/y, L)$$

depends only on ρ_y.
It is its correspondent in the conjectural p-adic Langlands correspondence for $\mathrm{GL}_2 K$.

Breuil–Herzig–Hu–Morra–Schraen

Theorem (BHHMS)

Suppose $K = \mathbf{Q}_{p^f}$ is unramified. Suppose R_∞ is formally smooth and genericity conditions on $\bar{r}$. Then:

1. *The Gelfand-Kirillov dimension of π is $\leq [K : \mathbf{Q}_p] = f$.*
2. *The Gelfand-Kirillov dimension of π is $[K : \mathbf{Q}_p] = f$.*
3. *The R_∞-module $\mathbb{M}_\infty$ is faithfully flat.*
4. *For each $y \in \operatorname{SpecMax} R_\infty[1/p]$, $\Pi_y \neq 0$.*

- (1) $\Longrightarrow$ (2) because Gee–Newton have proved "$\geq$".
- (2) $\Longrightarrow$ (3): "Miracle Flatness" due to Gee–Newton.
- (3) $\Longrightarrow$ (4): more or less direct.

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Direct application?

A result of DPS says that each Π_y has an inf. char.. Therefore, $\forall y$

$$\mathrm{GKdim}(\Pi_y) = \mathrm{GKdim}\left(\mathbb{M}_\infty[1/p]/y\right) \leq [K : \mathbf{Q}_p].$$

Question: is this enough to get $\mathrm{GKdim}(\pi) \leq [K : \mathbf{Q}_p]$ and consequently flatness?
Recall: let M be an f.g. p-torsion free $\mathcal{O}_L[\![H]\!]$-module, then

$$\mathrm{GKdim}(M/\varpi_L M) = \mathrm{GKdim}(M[1/p]).$$

Idea: take $M = \mathbb{M}_\infty/(y \cap R_\infty)$ (and $\mathbb{M}_\infty/(\varpi_L, y \cap R_\infty) = \pi^\vee$).
Problem: this only gives $\mathrm{GKdim}\left(M_{p\text{-t.f.}}/\varpi_L\right)$ not $\mathrm{GKdim}(\pi^\vee)$.
Observation: as soon as we find a y for which M has no p-torsion, we get (faithful) flatness of $\mathbb{M}_\infty$ (implying p-torsion freeness of M for all y a posteriori!).

p-adic Banach representations in families

Let G be a p-adic Lie group ($\mathrm{GL}_2 K$), H a compact open.
Let $(R, \mathfrak{m}_R)$ be a complete Noetherian local $\mathcal{O}_L$-algebra with residue field k_L.

Definition

A *family of (admissible) p-adic Banach representations of G* is a finitely generated $R[\![H]\!]$-module M with a compatible action of G without p-torsion.

Indeed, for every $y \in \mathrm{SpecMax}\, R[1/p]$ we get an admissible Banach rep. of G

$$\Pi_{M,y} := \mathrm{Hom}_L^{\mathrm{cont}}\left(M[1/p]/y, L\right).$$

Example

$(R, \mathfrak{m}_R) = (R_\infty, \mathfrak{m}_\infty)$, $M = \mathbb{M}_\infty$.

Infinitesimal character in families

DPS: there is a notion of "infinitesimal character in families" of M:

$$\text{“}\lambda\colon \mathcal{Z}\big(U(L\otimes_{\mathbf{Q}_p}\operatorname{Lie} G)\big)\to R^{\mathrm{rig}}\text{”}.$$

If M has such a λ, then each $\Pi_{M,y}$ has an infinitesimal character. What they actually proved:

Theorem (DPS)

The patched $R_\infty[\![\mathrm{GL}_2 K]\!]$-module $\mathbb{M}_\infty$ has an infinitesimal character in families.

Main theorem 2

Let M be a finitely generated p-torsion free $R[\![H]\!]$-module. Set

$$\pi_M := \mathrm{Hom}_{R_\infty}^{\mathrm{cont}}(M, k_L).$$

Example

$\pi_{\mathbb{M}_\infty} = \pi$.

Theorem (S.)

Suppose M has an infinitesimal character in families. Then

$$\mathrm{GKdim}(\pi_M) \leq [K : \mathbf{Q}_p].$$

Remark

Also holds for $D^\times$ instead of $\mathrm{GL}_2 K$.

Input to main theorem 2 (very vague)

- Can take $\mathcal{O}_L[\![X_1 \ldots, X_s]\!] \twoheadrightarrow R$, so wlog $R = \mathcal{O}_L[\![\mathbf{Z}_p^s]\!]$.
- Then M is an $\mathcal{O}_L[\![\mathbf{Z}_p^s \times H]\!]$-module.
- The inf. char. is $R^{\mathrm{rig}} = D(\mathbf{Z}_p^s, L)$-valued. This looks familiar!
- However... Schneider-Teitelbaum's theory of radius r distribution algebras $D_r(H, L)$ needs to be generalized to double radii $D_{r_1,r_2}(\mathbf{Z}_p^s \times H, L)$. Same for the filtrations.
- Then it gets quite technical...

Application to patching

The result of DPS ($\mathbb{M}_\infty$ has inf. char. in families) combined with ours now gives:

Corollary

The Gelfand-Kirillov dimension of $\pi = \pi_{\mathbb{M}_\infty}$ is $\leq [K : \mathbf{Q}_p]$.

Under some more restrictive hypotheses:

Corollary

Suppose R_∞ is formally smooth. Then:

1. *The Gelfand-Kirillov dimension of π is $[K : \mathbf{Q}_p]$.*
2. *The R_∞-module $\mathbb{M}_\infty$ is faithfully flat.*
3. *For each $y \in \operatorname{SpecMax} R_\infty[1/p]$, $\Pi_y \neq 0$.*

Variant of main theorem 2

In applications (such as completed cohomology), one might instead like to work with p-adic Lie groups such as

$$P\colon\ \mathrm{GL}_2 K_1 \times \cdots \times \mathrm{GL}_2 K_r \times D_1^\times \times \cdots \times D_s^\times, \tag{2}$$

with D_i the quaternion algebra over K_i' and all K_i, K_i' p-adic fields.

Example

Let F be a number field and $\mathbb{G} = \mathrm{Res}_{F/\mathbf{Q}} \mathrm{GL}_2$. Then $\mathbb{G}(\mathbf{Q}_p)$ looks like (2).

Theorem

Let $H \subset P$ be a compact open subgroup and M a finitely generated p-torsion free $R[\![H]\!]$-module. Suppose it has an infinitesimal character in families. Then

$$\mathrm{GKdim}(M/\mathfrak{m}_R) \leq [K_1\colon \mathbf{Q}_p] + \cdots + [K_r\colon \mathbf{Q}_p] + [K_1'\colon \mathbf{Q}_p] + \cdots + [K_s'\colon \mathbf{Q}_p].$$

Application to completed cohomology (to be worked out!)

Let F be a quadratic imaginary field in which p is split. Let $\mathbb{G} = \mathrm{Res}_{F/\mathbf{Q}} \mathrm{GL}_2 F$ and consider the completed cohomology $\widetilde{H}^\bullet_{\mathfrak{m}}$ of its locally symmetric space at a level $K^p \subset \mathrm{GL}_2(\mathbf{A}_F^{p,\infty})$ away from p localized at a "decomposed generic" and "non-Eisenstein" maximal ideal $\mathfrak{m}$ of the big Hecke algebra $\mathbf{T}^S(K^p)$.

- Each $\widetilde{H}^i_{\mathfrak{m}}$ is a family of Banach representations of $G(\mathbf{Q}_p)$.
- Ivančić–McDonald: each $\widetilde{H}^i_{\mathfrak{m}}$ has an inf. char. in families.
- Variant of main theorem 2: $\mathrm{GKdim}\left(\widetilde{H}^1_{\mathfrak{m}}/\mathfrak{m}\right) \leq [F : \mathbf{Q}]$.
- This establishes the hypothesis of Prop. 4.3.1 of Gee–Newton proving "big $R = \mathbf{T}$" in their setting.

Thank you for listening!